number of control grid points is used, difference between the second sensitivity components for the performance index ψ calculated with AVM and FDM is 11%. Note that even though the DDM and AVM have different sensitivity coefficients, the first variations agree almost 100%. Comparing (1-NIP) and RE for performance index ψ, we observe that the order of magnitudes is different. For example, the order of (1-NIP) for DDM is -8, and AVM is -4, and the order of RE for DDM is -4, and AVM is -2. This also is true for the sensitivity coefficients of the terminal constraint h. This indicates that DDM gives more accurate gradients than AVM for this application, as may be observed from the data given in Table 1.

A large number of control grid points ($NU = 51$) is also used to compare the sensitivity coefficients for all the constraints. Other data are the same as in the preceding case. Table 2 contains comparison of sensitivity coefficients for ψ, h, and ϕ. Because of the large amount of data, component by component comparison of the sensitivity coefficients is not included in the table. Note that $\mathrm{d}\psi/\mathrm{d}\boldsymbol{b}$ and $\mathrm{d}h/\mathrm{d}\boldsymbol{b}$ have 51 components, and $\mathrm{d}\phi/\mathrm{d}\boldsymbol{b}$ is a matrix of gradients of dimension 51×101 because there are 101 time grid points. Also, Table 2 does not contain comparisons of first variations because they were all very close to 100 with both the AVM and DDM. Also for ϕ, comparisons for only 11 out of 101 time grid points are shown in the table; the remaining points had similar results. It may be observed from the data in Table 2 that the accuracy of the analytical methods is good. However, DDM is more accurate than AVM for this example. This agrees with the results previously reported.[4] In addition, DDM is easier to implement in a computer program.

IV. Discussion and Conclusions

In this paper, the problem of verifying design sensitivity analysis when implemented into a computer program is addressed. This verification is quite important because incorrect gradients can cause the optimization process to fail or converge to nonoptimum points. Three verification methods are presented: the first variation, normalized inner product, and relative error in the largest component. It is observed that the first variation method is considerably less expensive computationally than the other two methods. In the method, all design variables are changed in $\Delta\boldsymbol{b}$ to evaluate the first variation by finite differences. However, in the other two methods, design variables must be changed one by one and functions re-evaluated to calculate the gradient vector by the finite-difference method.

The first variation method can measure only the global accuracy of sensitivity calculations. It is shown that even if global accuracy is good, there can be large errors in components of the sensitivity vector. Therefore, this method is not suitable for verifying design sensitivity analysis and is not recommended for general usage.

It is clear that if components of the sensitivity vector are good, then global accuracy is automatically good. Therefore, the best approach is to verify each component of the sensitivity vector. However, when there is a large difference in the magnitude of the calculated gradient components, the method can indicate large errors even if sensitivity calculations are correct. So, the approach should be used with caution and should be augmented with the normalized inner product or the relative error criterion. This procedure of verifying sensitivity calculations is recommended for general usage. However, since the procedure can be computationally expensive, it is recommended to first verify the sensitivity calculations on several small-scale problems before testing the program on large-scale applications.

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Mass Matrix Modification Using Element Correction Method

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Introduction

IN a dynamic analysis, the mode shapes obtained from vibration tests are generally not orthogonal to the analytical mass matrix. Several methods have been published which assume that the analytical mass matrix is correct and thereby modify the measured modes to achieve orthogonality.[1-4] An alternate approach is to assume that the measured mode shapes are exact and thus the analytical mass matrix is modified.[5-8]

The reason that the mass matrix correction combined with the Lagrange multiplier method has merit is that the resulting analytical model predicts the results obtained from experiments exactly. Also, this method is simpler and requires less computer time than other approaches. However, this method has limitations: 1) the banded character of the mass matrix is destroyed after the modification and the modified mass matrix does not represent the mass distribution of an actual structure, and 2) the total weight of the system is changed from the test value after the modification.

In this Note, an analytical mass matrix modification method via element correction is proposed based on an incomplete set of modal test data. This approach has the capability to change only those elements that require modification and, therefore, preserves a banded or a full mass matrix. Furthermore, the total weight constraint is also enforced in the analysis. It is felt that the present method is a viable technique for improving an analytical model based on an incomplete set of test data.

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Technical Approach

In an incomplete test, the measured modal matrix $\Phi(n \times m)$ is rectangular, where $n > m$, and the analytical mass matrix $M_A(n \times n)$ is a symmetric matrix. The measured modes have been normalized to fulfill the basic orthogonality requirement:

$$\Phi^T M \Phi = I \tag{1}$$

where M $(n \times n)$ is a symmetric matrix and represents a corrected mass matrix. In addition to Eq. (1), a corrected mass matrix has to satisfy the total weight constraint after modification, as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} U_{ij} m_{ij} = W \tag{2}$$

where U_{ij} is the weight per unit mass, W the total weight of the system, and m_{ij} the ith row and jth column element of the mass matrix M.

The simplest way to correct a mass matrix satisfying Eqs. (1) and (2) is given by minimizing a weighted Euclidean norm:

$$\epsilon = \frac{1}{2} \left\| M_A^{-1/2} (M - M_A) M_A^{-1/2} \right\|$$
$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sum_{q=1}^{n} m_{A_{iq}}^{-1/2} \sum_{p=1}^{n} (m_{qp} - m_{A_{qp}}) m_{A_{pj}}^{-1/2} \right]^2 \tag{3}$$

Using Lagrange multipliers to include the constraints of Eqs. (1) and (2), the Lagrange function Ψ is defined as

$$\Psi = \epsilon + |\Lambda(\Phi^T M \Phi - I)| + \beta \left(\sum_{i=1}^{n} \sum_{j=1}^{n} U_{ij} m_{ij} - W \right) \tag{4}$$

where

$$|\Lambda(\Phi^T M \Phi - I)| = \sum_{p=1}^{m} \sum_{q=1}^{m} \lambda_{pq} \left(\sum_{i=1}^{n} \Phi_{pi} \sum_{j=1}^{n} m_{ij} \Phi_{jq} - \delta_{pq} \right) \tag{5}$$

For convenience, the matrix $\Lambda(m \times m)$ is assumed to be a symmetric Lagrange multiplier for Eq. (1). (Note that, if Λ is not a symmetric matrix, then $M = M^T$ symmetric matrix constraint is needed to yield the same result.) Also, β is a scalar Lagrange multiplier for constraint [Eq. (2)], and δ_{pq} is the Kronecker delta.

The partial differentiation of Ψ with respect to m_{ij} is set to zero, resulting in the m_{ij} for minimum Ψ. In matrix form, this becomes

$$\left[\frac{\partial \Psi}{\partial m_{ij}} \right] = M_A^{-1} (M - M_A) M_A^{-1} + \Phi \Lambda \Phi^T + \beta U = 0 \tag{6}$$

Rearranging Eq. (6) gives

$$M = M_A - M_A \Phi \Lambda \Phi^T M_A - \beta M_A U M_A \tag{7}$$

Since M is an $n \times n$ symmetric matrix, Eq. (7) can be represented as a column vector containing all of the diagonal and right upper off-diagonal elements. This technique has the capability of choosing the elements of the mass matrix required to be modified. That is,

$$\{m_{ij}\} = \{m_{A_{ij}}\} - [A]\{\lambda_{pq}\} - \beta\{B_{ij}\} \tag{8}$$

where $\{m_{ij}\}$ and $\{m_{A_{ij}}\}$ are column vectors with a maximum number of elements up to $\frac{1}{2}n(n+1)$ elements. Both column vectors include all of the diagonal and right upper off-diagonal elements of the matrices M and M_A, respectively. Also, $[A]\{\lambda_{pq}\}$ and $\{B_{ij}\}$ represent similar terms of the matrices $M_A \Phi \Lambda \Phi^T M_A$ and $M_A U M_A$, respectively. Here, $[A]$ is a $\frac{1}{2}n(n+1)$ row by $\frac{1}{2}m(m+1)$ column rectangular matrix, $\{\lambda_{pq}\}$ a vector with $\frac{1}{2}m(m+1)$ elements, and $\{B_{ij}\}$ a vector with $\frac{1}{2}n(n+1)$ elements. Matrix $[A]$ can be expressed as

$$[A]\{\lambda_{pq}\} = \left[\sum_{i=j=1}^{n} \left(\sum_{S=1}^{n} m_{A_{is}} \Phi_{sp} \right) \left(\sum_{t=1}^{n} m_{A_{it}} \Phi_{tq} \right) + \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \left(\sum_{S=1}^{n} m_{A_{is}} \Phi_{sp} \right) \left(\sum_{t=1}^{n} m_{A_{jt}} \Phi_{tq} \right) + \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \left(\sum_{S=1}^{n} m_{A_{is}} \Phi_{sq} \right) \left(\sum_{t=1}^{n} m_{A_{jt}} \Phi_{tp} \right) \right] \{\lambda_{pq}\} \tag{9}$$

where

if $p = q$,	then	$p = 1$ to m
$p \neq q$,	then	$p = 1$ to $(m-1)$
		$q = (p+1)$ to m

Similarly, column vector $\{B_{ij}\}$ can be expressed as

$$\{B_{ij}\} = \sum_{S=1}^{n} m_{A_{is}} \sum_{t=1}^{n} U_{st}\, m_{A_{tj}} \tag{10}$$

where

if $i = j$,	then	$i = 1$ to n
$i \neq j$,	then	$i = 1$ to $(n-1)$
		$j = (i+1)$ to n

From Eq. (1), the orthogonality constraint equation expressed in element form gives

$$[G]\{m_{ij}\} = \{\delta_{pq}\} \tag{11}$$

where $\{\delta_{pq}\}$ is a vector with $\frac{1}{2}m(m+1)$ elements, and $[G]$ is a $\frac{1}{2}m(m+1) \times \frac{1}{2}n(n+1)$ rectangular matrix and can be obtained from

$$[G]\{m_{ij}\} = \left[\sum_{i=j=1}^{n} \Phi_{ip} \Phi_{jq} + \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \Phi_{ip} \Phi_{jq} + \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \Phi_{jp} \Phi_{iq} \right] \{m_{ij}\} \tag{12}$$

where

if $i = j$,	then	$i = 1$ to n
$i \neq j$,	then	$i = 1$ to $(n-1)$
		$j = (i+1)$ to n
if $p = q$,	then	$p = 1$ to m
$p \neq q$,	then	$p = 1$ to $(m-1)$
		$q = (p+1)$ to m

From Eq. (2), the total dynamic system weight constraint equation in element form can be shown to be

$$\langle U_{ij} \rangle \{m_{ij}\} = W \tag{13}$$

where $\langle U_{ij} \rangle$ is a row vector with $\frac{1}{2}n(n+1)$ elements, and W is a scalar. Substituting Eq. (8) into Eq. (11) yields

$$[G]\{m_{A_{ij}}\} - [G][A]\{\lambda_{pq}\} - \beta[G]\{B_{ij}\} = \{\delta_{pq}\} \tag{14}$$

Because $[G][A]$ is a $\frac{1}{2}m(m+1) \times \frac{1}{2}m(m+1)$ nonsingular matrix, the inverse matrix $[H]^{-1}$ exists, where

$$[H] = [G][A] \tag{15}$$

Combining Eqs. (14) and (15) allows the determination of $\{\lambda_{pq}\}$ as

$$\{\lambda_{pq}\} = [H]^{-1}[G]\{m_{A_{ij}}\} - \beta[H]^{-1}[G]\{B_{ij}\} - [H]^{-1}\{\delta_{pq}\} \tag{16}$$

Furthermore, from Eqs. (8) and (16), $\{m_{ij}\}$ is found as

$$\{m_{ij}\} = \{m_{A_{ij}}\} - [A][H]^{-1}[G]\{m_{A_{ij}}\} + [A][H]^{-1}\{\delta_{pq}\} + \beta[A][H]^{-1}[G]\{B_{ij}\} - \beta\{B_{ij}\} \tag{17}$$

Substituting Eq. (17) into Eq. (13) yields the scalar Lagrange multiplier β,

$$\beta = E/F \tag{18}$$

where

$$E = W - \langle U_{ij}\rangle\{m_{A_{ij}}\} + \langle U_{ij}\rangle[A][H]^{-1}[G]\{m_{A_{ij}}\} - \langle U_{ij}\rangle[A][H]^{-1}\{\delta_{pq}\} \tag{19}$$

$$F = \langle U_{ij}\rangle[A][H]^{-1}[G]\{B_{ij}\} - \langle U_{ij}\rangle\{B_{ij}\} \tag{20}$$

Therefore, from Eq. (17), the corrected mass matrix, satisfying both the orthogonality requirement and total weight constraints, is obtained using the element modification method as

$$\{m_{ij}\} = \{m_{A_{ij}}\} - [A][H]^{-1}[G]\{m_{A_{ij}}\} + [A][H]^{-1}\{\delta_{pq}\} + (E/F)[A][H]^{-1}[G]\{B_{ij}\} - (E/F)\{B_{ij}\} \tag{21}$$

Conclusions

An analytical mass matrix modification has been achieved using the element correction method. This method preserves the original characteristics of a mass matrix while enforcing both orthogonality and total weight constraints. In this approach, only the mass elements that require correction are modified, and therefore the modified matrix is banded or a full matrix. Also, no iteration is required in order to ensure that the mass matrix satisfies the total weight constraint. This method is very useful to improve the analytical mass model based on an incomplete modal test.

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